Faculty of Engineering and Technology

Discrete Mathematics (CSPS-111)

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Outlines

- Basic Set Relations
- The Empty Set
- Sets Are Objects
- Cardinality of Sets
- The Power Set
- Cartesian Products of Sets
- Set Operations

Basic Set Relations

- $X \in S$ ("*x* is in *S*") is the proposition that object *x* is an *element* or *member* of set *S*.
  - *e.g.* $3 \in \mathbf{N}$, "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
- Can define set equality in terms of $\in$ relation:

  $\forall S,T: S=T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$

  "Two sets are equal **iff** they have all the same members."
- $X \notin S :\equiv \neg(x \in S)$ "*x* is not in *S*"

The Empty Set

- $\varnothing$ (“null”, “the empty set”) is the unique set that contains no elements whatsoever.
- $\varnothing = \{\} = \{x|\textbf{False}\}$
- No matter the domain of discourse,

  we have the axiom

  $$\neg\exists x\colon x\in\varnothing.$$

Subset and Superset Relations

- $S\subseteq T$ (“S is a subset of T”) means that every element of S is also an element of T.
- $S\subseteq T \Leftrightarrow \forall x\,(x\in S \rightarrow x\in T)$
- $\varnothing\subseteq S$, $S\subseteq S$.
- $S\supseteq T$ (“S is a superset of T”) means $T\subseteq S$.
- Note $S=T \Leftrightarrow S\subseteq T\wedge S\supseteq T$.
- means $\neg(S\subseteq T)$, i.e. $\exists x(x\in S \wedge x\notin T)$

Sets Are Objects

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S=\{x \mid x \subseteq \{1,2,3\}\}$

  then $S=\{\emptyset,$

  $\{1\}, \{2\}, \{3\},$

  $\{1,2\}, \{1,3\}, \{2,3\},$

  $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality of Sets

- $|S|$ (read "the *cardinality* of *S*") is a measure of how many different elements *S* has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,

  $|\{\{1,2,3\},\{4,5\}\}| =$ 2
- We say *S* is *infinite* if it is not *finite*.
- What are some infinite sets we've seen?
- If a set S contains n distinct elements, $n \in N$, we call S a finite set with cardinality n.

Examples:

- A = {Mercedes, BMW, Porsche}, $|A| = 3$
- $B = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|B| = 4$
- $C = \varnothing$ $|C| = 0$
- $E = \{ x \in N \mid x \geq 7000 \}$ E is infinite!

The Power Set

- The *power set* P(*S*) of a set *S* is the set of all subsets of *S*. P(*S*) = $\{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Sometimes P(*S*) is written $\mathbf{2}^S$.

  Note that for finite *S*, $|P(S)| = 2^{|S|}$.
- It turns out that |P(**N**)| > |**N**|.

  There are different sizes of infinite sets!

- P(A) "power set of A" (also written as 2^A)
- P(A) = $\{B \mid B \subseteq A\}$ (contains all subsets of A)

Examples:

- A = {x, y, z}
- P(A) = {∅, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}}

- A = ∅
- P(A) = {∅}

  Note: |A| = 0, |P(A)| = 1

The Power Set

- Cardinality of power sets: | P(A) | = $2^{|A|}$
- Imagine each element in A has an “on/off” switch
- Each possible switch configuration in A corresponds to one subset of A, thus one element in P(A)

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are 2 × 2 × 2 = 8 elements in P(A)

Cartesian Products of Sets

- For sets A, B, their *Cartesian product*

  $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.

- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$

- Note that for finite A, B, $|A \times B| = |A||B|$.

- Note that the Cartesian product is ***not*** commutative: $\neg\forall AB: A \times B = B \times A$.

- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

•The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of n objects.

•Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and

$(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e.

$a_i = b_i$ for $1 \leq i \leq n$.

•The Cartesian product of two sets is defined as:

•$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Cartesian Product

- Example:
- A = {good, bad}, B = {student, prof}
- A×B = {

(good, student), (good, prof), (bad, student), (bad, prof)}

B x A ={ (student, good), (prof, good), (student, bad), (prof, bad)}

Example: A = {x, y}, B = {a, b, c}

A x B = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

- The Cartesian product of two or more sets is defined as:
- $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Set Operations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: A = {a, b}, B = {b, c, d}

- $A \cup B = \{a, b, c, d\}$

- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Example: A = {a, b}, B = {b, c, d}

- $A \cap B = \{b\}$
- Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$

•Two sets are called disjoint if their intersection is empty, that is, they share no elements:

•$A \cap B = \emptyset$

•The difference between two sets A and B contains exactly those elements of A that are not in B:

•$A-B = \{x \mid x \in A \wedge x \notin B\}$

Example: A = {a, b}, B = {b, c, d}, A-B = {a}

•Cardinality: $|A-B| = |A| - |A \cap B|$