

Cauchy Riemann equations:

These equations, in hydrodynamics, are sometimes called "Cauchy Riemann equations".

Relation between Stream Function and Velocity Potential

$$\left. \begin{aligned} u &= -\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \right\}$$

- a stream function is that the difference of its values at two points
- represent the flow across any line joining the points. Thus if two lie on the same stream line, then
- there being no flow across a stream line, the difference between the stream functions ψ_1 and $\psi_2 = 0$;
- this means the streamline is given by:
- $\psi = \text{constant}$.
- Similarly, $\phi = \text{constant}$, represents a case for which the velocity potential is same at every point, and hence it represents an equipotential line.
- Let, two curves $\phi = \text{constant}$ and $\psi = \text{constant}$ intersect each other at any point. At the point of
- intersection the slopes are:

For the curve $\phi = \text{constant}$: Slope = $\frac{\partial y}{\partial x} = \frac{\left(\frac{\partial \phi}{\partial x}\right)}{\left(\frac{\partial \phi}{\partial y}\right)} = \frac{-u}{-v} = \frac{u}{v}$

For the curve $\psi = \text{constant}$: Slope = $\frac{\partial y}{\partial x} = \frac{\left(\frac{\partial \psi}{\partial x}\right)}{\left(\frac{\partial \psi}{\partial y}\right)} = \frac{-v}{+u} = -\frac{v}{u}$

Now, product of the slopes of these curves

$$= \frac{u}{v} \times -\frac{v}{u} = -1$$

- It shows that these two sets of curves, viz stream lines and equipotential lines intersect each other orthogonally at all points of intersection.

Fluid Properties

- Verify whether the following functions are valid potential functions:
- (i) $\phi = A(x^2 - y^2)$ (ii) $\phi = A \cos x$

Solution. (i) $\phi = A(x^2 - y^2)$:

$$\frac{\partial \phi}{\partial x} = 2Ax; \quad \frac{\partial \phi}{\partial y} = -2Ay$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2A; \quad \frac{\partial^2 \phi}{\partial y^2} = -2A$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2A + (-2A) = 0$$

Hence, $\phi = A(x^2 - y^2)$ is a valid potential function (Ans.)

(ii) $\phi = A \cos x$:

$$\frac{\partial \phi}{\partial x} = -A \sin x; \quad \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = -A \cos x; \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -A \cos x \neq 0$$

Hence, $\phi = A \cos x$ is not a valid function (Ans.)

Fluid Properties

- Which of the following functions represent possible irrotational flow?

(i) $\psi = A(x^2 - y^2)$

(ii) $\psi = xy$

(iii) $\phi = \left(r - \frac{2}{r}\right) \sin \theta$

(iv) $\phi = Ur \cos \theta + \frac{U}{r} \cos \theta$

Solution. For an irrotational fluid flow phenomenon ϕ as well ψ satisfy Laplace equation.

(i) $\psi = A(x^2 - y^2)$:

$$\frac{\partial \psi}{\partial x} = 2Ax; \quad \frac{\partial \psi}{\partial y} = -2Ay$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2A; \quad \frac{\partial^2 \psi}{\partial y^2} = -2A$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2A - 2A = 0$$

Hence, $\psi = A(x^2 - y^2)$ represents a possible irrotational flow (Ans.)

(ii) $\psi = xy$:

$$\frac{\partial \psi}{\partial x} = y; \quad \frac{\partial \psi}{\partial y} = x$$

$$\frac{\partial^2 \psi}{\partial x^2} = 0; \quad \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Hence, $\psi = xy$ represents a possible irrotational flow. (Ans.)



Fluid Properties

$$(iii) \phi = \left(r - \frac{2}{r}\right) \sin \theta :$$

Laplace equation in radial coordinates (r, θ) is given as:

$$\frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\frac{\partial \phi}{\partial r} = \left(1 + \frac{2}{r^2}\right) \sin \theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = -\frac{2}{r^3} \sin \theta$$

$$\frac{\partial \phi}{\partial \theta} = \left(r - \frac{2}{r}\right) \cos \theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = -\left(r - \frac{2}{r}\right) \sin \theta = \left(\frac{2}{r} - r\right) \sin \theta$$

L.H.S. of Laplace equation is:

$$\begin{aligned} & \frac{1}{r} \times \left(1 + \frac{2}{r^2}\right) \sin \theta - \frac{2}{r^3} \sin \theta + \frac{1}{r^2} \left(\frac{2}{r} - r\right) \sin \theta \\ &= \sin \theta \left(\frac{1}{r} + \frac{2}{r^3} - \frac{2}{r^3} + \frac{2}{r^3} - \frac{1}{r}\right) \\ &= \sin \theta \left(\frac{2}{r^3}\right) \neq 0 \end{aligned}$$

Hence, the given function doesnot represent any possible irrotational flow. (Ans.)

Fluid Properties

- The velocity components in a fluid flow are given by: $u = 2xy$; $v = a^2 + x^2 - y^2$
- (i) Show that the flow is possible. (ii) Derive the relative stream function

Solution. Given: $u = 2xy$; $v = a^2 + x^2 - y^2$...Velocity components

$$(i) \quad \frac{\partial u}{\partial x} = 2y; \quad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2y + (-2y) = 0$$

The continuity equation for steady, incompressible flow is satisfied.

Hence, flow is possible. (Ans.)

(ii) The stream function ψ is related to u and v as:

$$u = \frac{\partial \psi}{\partial y} = 2xy$$

or, $\psi = \int 2xy \, dy = xy^2 + f(x)$... (i)

$$-\frac{\partial \psi}{\partial x} = -y^2 - f'(x) = v = a^2 + x^2 - y^2$$

Hence, $f'(x) = -(a^2 + x^2)$

or, $f(x) = -a^2x - \frac{x^3}{3} + \text{constant}$

Inserting for $f(x)$ in eqn. (i) we get:

$$\psi = xy^2 - ax^2 - \frac{x^3}{3} + \text{constant}$$

Thus, the relative $\psi = xy^2 - a^2x - \frac{x^3}{3} + \text{constant}$ (Ans.)

Fluid Properties

- A flow is described by the stream function $\psi = 4xy$. Locate the point at which the velocity vector has a magnitude 7 units and makes an angle of 150° with X-axis.

Solution. Stream function, $\psi = 4xy$

...Given

The velocity components for the given flow field are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(4xy) = 4x$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(4xy) = -4y$$

$$V = \sqrt{u^2 + v^2} \quad \text{or} \quad 7 = \sqrt{(4x)^2 + (-4y)^2} = 4\sqrt{x^2 + y^2} \quad \dots(i)$$

Also, $\tan \theta = \frac{v}{u} \quad \text{or} \quad \tan 150^\circ = \frac{-4y}{4x} = -\frac{y}{x}$

or, $-0.577 = -\frac{y}{x} \quad \text{or} \quad y = 0.577x$

Substituting for y in eqn. (i), we get:

$$7 = 4\sqrt{x^2 + (0.577x)^2} = 4.62x$$

$\therefore x = \frac{7}{4.62} = 1.515 \text{ (Ans.)}$