# Fluid Properties

- Similarly, work done by the pressure at MM in moving the liquid to M′M′ = – p2.A2 . dl2
- (– ve sign indicates that direction of p2 is opposite to that of p1)
- ∴ Total work done by the pressure
- = p1 . A1 dl1 – p2 A2 dl2
- = p1 . A1 dl1 – p2 A1 dl1 (? A1dl1 = A2dl2)
- = A1 . dl1 (p1 – p2)

$$= \frac{W}{w}(p_1 - p_2) \qquad \left(\because A_1 . dl_1 = \frac{W}{w}\right)$$

$$\text{Loss of potential energy} = W(z_1 - z_2)$$

$$\text{Gain in kinetic energy} = W\left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right) = \frac{W}{2g}(V_2^2 - V_1^2)$$

Also, Loss of potential energy + work done by pressure = Gain in kinetic energy

$$\therefore W(z_1 - z_2) + \frac{W}{w}(p_1 - p_2) = \frac{W}{2g}(V_2^2 - V_1^2)$$

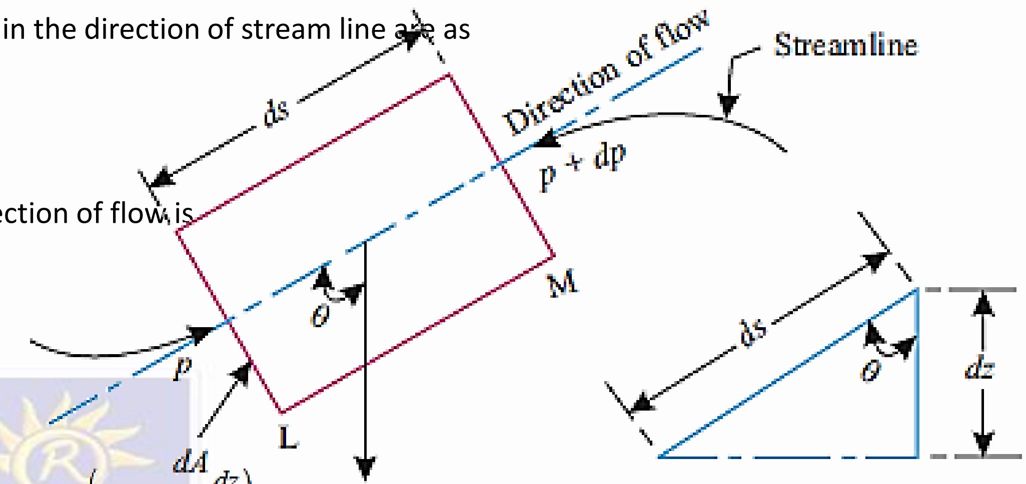
or, $$(z_1 - z_2) + \left(\frac{p_1}{w} - \frac{p_2}{w}\right) = \left(\frac{V_2^2}{2g} - \frac{V_1^2}{2g}\right)$$

or, $$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

which proves *Bernoulli's equation.*

- which proves Bernoulli's equation.
- Assumptions:
- It may be mentioned that the following assumptions are made in the derivation of Bernoulli's
- equation:
- 1. The liquid is ideal and incompressible.
- 2. The flow is steady and continuous.
- 3. The flow is along the stream line, i.e., it is one-dimensional.
- 4. The velocity is uniform over the section and is equal to the mean velocity.
- 5. The only forces acting on the fluid are the gravity forces and the pressure forces.

- Consider steady flow of an ideal fluid along the stream tube. Separate out a small element of fluid of cross-sectional area dA and length ds from stream tube as a free body from the moving fluid. Fig. shows such a small element LM of fluid of cross-section area dA and length ds.
- Let, p = Pressure on the element at L, p + dp = Pressure on the element at M, and V = Velocity of the fluid element.
- The external forces tending to accelerate the fluid element in the direction of stream line are as
- follows:
- 1. Net pressure force in the direction of flow is,
- p.dA – (p + dp) dA = – dp . dA ...(i)
- 2. Component of the weight of the fluid element in the direction of flow is
- = – ρ.g.dA.ds. cosθ




$$= -\rho g . dA . ds \left(\frac{dz}{ds}\right) \qquad \left(\because \cos\theta = \frac{dz}{ds}\right)$$

$$= -\rho . g . dA . dz \qquad ...(ii)$$

Mass of the fluid element $= \rho . dA . ds$ ...(iii)

The acceleration of the fluid element

$$a = \frac{dV}{dt} = \frac{dV}{ds} \times \frac{ds}{dt} = V . \frac{dV}{ds} \qquad ...(iv)$$

w, according to Newton's second law of motion, Force = Mass × acceleration

$$\therefore \qquad -dp . dA - \rho . g . dA . dz = \rho . dA . ds \times V \frac{dV}{ds}$$

Dividing both sides by $\rho . dA$, we get:

$$\frac{-dp}{\rho} - g . dz = V . dV$$

or, $$\frac{dp}{\rho} + V . dV + g . dz = 0 \qquad ...(6.3)$$

This is the required **Euler's equation** for motion, and is in the form of *differential equation.*

Integrating the above eqn., we get:

# Fluid Properties

$$\frac{1}{\rho}\int dp + \int V.dV + \int g.dz = \text{constant}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Dividing by g, we get:

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$

or,
$$\frac{p}{w} + \frac{V^2}{2g} + z = \text{constant}$$

or, in other words,

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

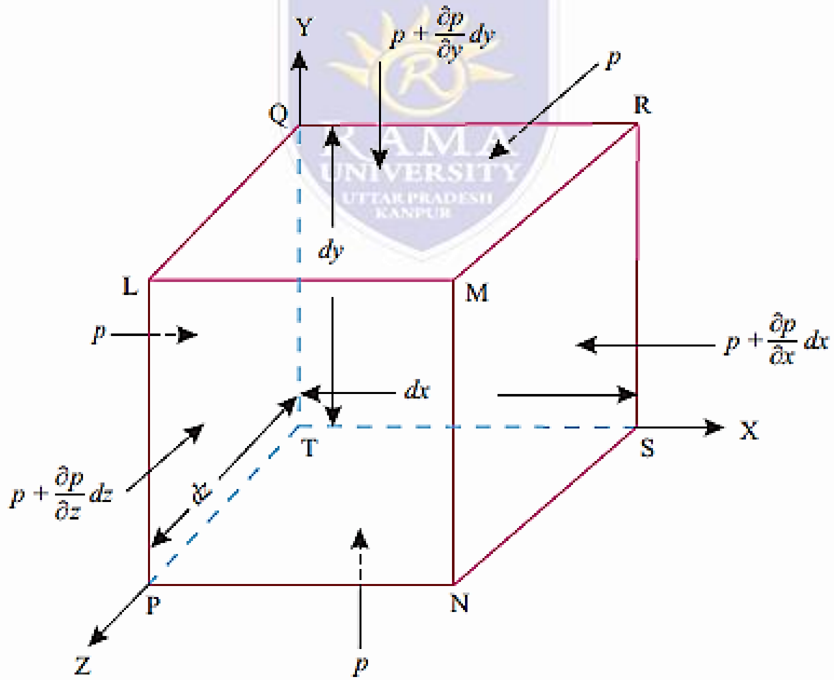
which proves *Bernoulli's equation*.

# Fluid Properties

- Euler's equation in Cartesian coordinates:
- Consider an infinitely small mass of fluid enclosed in an elementary parallelopiped of sides dx, dy and dz as shown in Fig. 6.3. The motion of the fluid element is influenced by the following forces:
- (i) Normal forces due to pressure: The intensities of hydrostatic pressure acting normal to each face of the parallelepiped are
- shown in Fig. The net pressure force in the X-direction

$$= p \,.\, dy \,.\, dz - \left( p + \frac{\partial p}{\partial x} dx \right) dy\, dz$$

- ii) Gravity or body force: $= -\frac{\partial p}{\partial x} dx \,.\, dy \,.\, dz$
- Let B be the body force per unit mass of fluid having components Bx , By and Bz in the X, Y and Z directions respectively.
- Then, the body force acting on the parallelopiped in the direction of X-coordinate is = Bx .ρ.dx .dy. dz.



**(iii) Inertia forces:**

The inertia force acting on the fluid mass, along the $X$-coordinate is given by,

$$\text{Mass} \times \text{acceleration} = \rho .\, dx .\, dy.\, dz.\ \frac{du}{dt}$$

As per Newton's second law of motion summation of forces acting in the fluid element in any direction equals the resulting inertia forces in that direction. Thus, along X-direction:

$$B_x .\rho.\, dx.dy.dz - \frac{\partial p}{\partial x}\ dx.dy.dz = \rho.dx.dy.dz.\frac{du}{dt}$$

Dividing both sides by $\rho.dx.dy.dz$, we have:

$$B_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{du}{dt} \qquad ...(i)$$

In this equation each term has dimensions of force per unit mass or acceleration. Obviously the total acceleration in a given direction is prescribed by the algebraic sum of the body force and the pressure gradient in that direction since the velocity components are functions of position and time, i.e., u = f(x, y, z, t), therefore, the total derivative of velocity u in the X-direction can be written as:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w; \text{ we have:}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Combining eqns. (*i*) and (*ii*), we get the force components as:

$$B_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Similarly,

$$B_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and,

$$B_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

For steady flow:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$$

Thus, the Euler's equation for a steady three-dimensional flow can be written as:

$$B_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$B_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$B_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

- In Euler's equation each term represents force per unit mass. Thus, if each equation is multiplied by the respective projections of the elementary displacement, the resulting equation would represent energy. Thus, in order to get total energy in the three-dimensional-steady-incompressible flow, the energy terms can be combined as follows:

$$B_x dx - \frac{1}{\rho}\frac{\partial p}{\partial x}dx = u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx$$

$$B_y dy - \frac{1}{\rho}\frac{\partial p}{\partial y}dy = u\frac{\partial v}{\partial x}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dy$$

$$B_z dz - \frac{1}{\rho}\frac{\partial p}{\partial y}dz = u\frac{\partial w}{\partial x}dz + v\frac{\partial w}{\partial y}dz + w\frac{\partial w}{\partial z}dz$$

From the equation of a stream line in a three-dimensional flow, we have:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$udy = vdx;\ vdz = wdy;\ udz = wdx$$

Substituting these values in eqns. (ix), (x) and (xi), we get:

$$B_x dx - \frac{1}{\rho}\frac{\partial p}{\partial x}dx = u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz$$

$$B_y dy - \frac{1}{\rho}\frac{\partial p}{\partial y}dy = v\frac{\partial v}{\partial x}dx + v\frac{\partial v}{\partial y}dy + v\frac{\partial v}{\partial z}dz \quad \text{...(xiii)}$$

$$B_z dz - \frac{1}{\rho}\frac{\partial p}{\partial z}dz = w\frac{\partial w}{\partial x}dx + w\frac{\partial w}{\partial y}dy + w\frac{\partial w}{\partial z}dz \quad \text{...(xiv)}$$

Acceleration terms are of form $u\frac{\partial u}{\partial x}$ which can be replaced by $\frac{1}{2}\frac{\partial(u^2)}{\partial x}$. Thus,

$$B_x dx - \frac{1}{\rho}\frac{\partial p}{\partial x}dx = \frac{1}{2}\left[\frac{\partial}{\partial x}(u^2)dx + \frac{\partial}{\partial y}(u^2)dy + \frac{\partial}{\partial z}(z^2)dz\right] = \frac{1}{2}d(u^2) \quad \text{...(xv)}$$

Similarly,

$$B_y dy - \frac{1}{\rho}\frac{\partial p}{\partial y}dy = \frac{1}{2}d(v^2) \quad \text{...(xvi)}$$

and,

$$B_z dz - \frac{1}{\rho}\frac{\partial p}{\partial y}dz = \frac{1}{2}d(w^2) \quad \text{...(xvii)}$$

where, V = Total velocity vector.
When gravity is the only body force acting on the third element, then:
Bx = 0, Bz = 0 and By = – g
By = – g since the gravitational force acts in the downward direction which is negative 'with' respect to Y, which is positive upward. Inserting these values in (xviii), we get:

$$-g - \frac{1}{\rho}dp = \frac{1}{2}d(V^2)$$

or,

$$-g - \frac{1}{\rho}dp = VdV$$

or,

$$\frac{dp}{\rho} + VdV + g = 0$$ which is the same as Euler's