Also, the rate of change of fluid mass in the element LMST

$$= \frac{\partial}{\partial t} \text{(Density} \times \text{Volume)}\, dt$$

$$= \frac{\partial}{\partial t}\left[\rho \times \frac{r d\theta + (r + dr) d\theta}{2} dr\right] dt \simeq \frac{\partial}{\partial t} (\rho r d\theta\, dr)\, dt$$

As per law of conservation of mass:

The total gain in mass = The rate of change of fluid mass in the element LMST

$$\therefore \qquad -\left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) r + \frac{\partial}{\partial \theta}(\rho v_\theta)\right] dr\, d\theta\, dt = \frac{\partial}{\partial t}(\rho r\, d\theta\, dr\, dt)$$

$$\text{or,} \qquad \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) r + \frac{\partial}{\partial \theta}(\rho v_\theta)\right] dr\, d\theta + \frac{\partial}{\partial t}(\rho r\, d\theta\, dr) = 0$$

For steady and compressible flow, $\frac{\partial}{\partial t} \rho r d\theta dr = 0$

$$\therefore \qquad \left[\rho v_r + \frac{\partial}{\partial r}(\rho v_r) r + \frac{\partial}{\partial \theta}(\rho v_\theta)\right] dr\, d\theta = 0$$

Further, for *incompressible flow*, ρ = constant.

$$\therefore \qquad v_r + \frac{\partial}{\partial r}(v_r) r + \frac{\partial}{\partial \theta}(v_\theta)$$

$$\text{or,} \qquad \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r \partial \theta} = 0$$

- In three-dimensional incompressible third flow, the velocity components in xand y-directions are: u = x2 + y2z3; v = – (xy + yz + zx) Use continuity equation to evaluate an expression for the velocity component w in the z-direction.

Solution. The continuity equation for a steady, three-dimensional incompressible fluid flow is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \qquad ...(i)$$

$$u = x^2+y^2z^3;\ v=-(xy+yz+zx)$$

$$\frac{\partial u}{\partial x} = 2x\ ;\ \frac{\partial v}{\partial y} = -(x+z)$$

Substituting these values in eqn. (*i*), we get:

$$2x-(x+z)+\frac{\partial w}{\partial z} = 0$$

or,

$$\frac{\partial w}{\partial z} = -x+z$$

Integrating *w.r.t.z* we have:

$$w = -xz+\frac{z^2}{2}+C$$

where *C* is a constant of integration which should be independent of *z* but may be function of *x* and/or *y i.e.* $C=f(x,y)$

$$\therefore \qquad w = -x+\frac{z^2}{2}+f(x,y) \text{ (Ans.)}$$

- The velocity components in x and y directions are given as u = 2xy3/3 –x2y andv = xy2 – 2yx3/3. Indicate whether the given velocity distribution is: (i) A possible field of flow; (ii) Not a possible field of flow

Solution. *Given.* $u = 2xy^3/3 - x^2y$, $v = xy^2 - 2yx^3/3$ ...Velocity components

A possible flow field (two-dimensional) must satisfy the continuity equation.

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{...(i)}$$

Now,
$$\frac{\partial u}{\partial x} = \frac{2}{3}y^3 - 2xy, \quad \frac{\partial v}{\partial y} = 2xy - \frac{2}{3}x^3$$

Substituting these values in eqn. (*i*), we get:

$$\left(\frac{2}{3}y^3 - 2xy\right) + \left(2xy - \frac{2}{3}x^3\right) = \frac{2}{3}(y^3 - x^3)$$

Since the continuity equation is *not* satisfied, the given velocity components, therefore, do not represent a possible flow field. (Ans.)

- A two-dimensional incompressible flow in cylindrical polar coordinates is givenby: vr = 2r sin θ cos θ; vθ = – 2r sin2θ, Determine whether these velocity components represent a physically possible flow field

Solution. The continuity equation for a steady, two-dimensional incompressible flow is

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{r\partial\theta} = 0 \qquad \text{...[Eqn. (5.28)]}$$

From the given velocity components, we have:

$$\frac{\partial v_r}{\partial r} = \frac{\partial}{\partial r}(2r \sin\theta\cos\theta) = 2\sin\theta\cos\theta$$

$$\frac{\partial v_\theta}{\partial\theta} = \frac{\partial}{\partial\theta}(-2r\sin^2\theta) = -4r\sin\theta\cos\theta$$

Inserting these values in the above equation, we get:

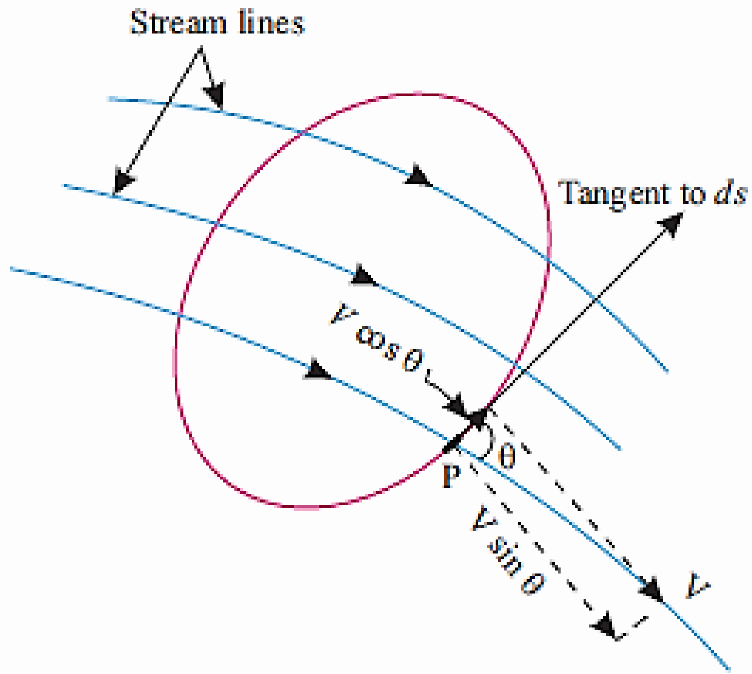
$$\frac{2r\sin\theta\cos\theta}{r} + 2\sin\theta\cos\theta - \frac{4r\sin\theta\cos\theta}{r} = 0$$

or $$2\sin\theta\cos\theta + 2\sin\theta\cos\theta - 4\sin\theta\cos\theta = 0$$

i.e., $L.H.S. = 0$

Thus the continuity equation is satisfied and hence the **flow is physically possible. (Ans.)**

Let us consider a closed curve in a twodimensional flow field shown in Fig. 5.18; the curve being cut by the stream lines. Let *P* be the point of intersection of the curve with one stream line, θ be the angle which the stream line makes with the curve. The component of velocity along the closed curve at the point of intersection is equal to *V* cos θ. **Circulation** Γ *is defined mathematically as the line integral of the tangential velocity about a closed path* (*contour*). Thus,



$$\Gamma = \oint V \cos\theta . ds$$

where, V = Velocity in the flow field at the element ds, and

θ = Angle between V and tangent to the path (in the positive anticlockwise directionalong the path) at that point

Circulation around regular curves can be obtained by integration. Let us consider the circulation around an elementary box (fluid element ABCD) shown in Fig. Starting from A and proceeding anticlockwise, we have:

$$d\Gamma = u\,\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x \right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y \right)\Delta x - v\,\Delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\Delta x . \Delta y$$

The vorticity (Ω) is defined as the *circulation per unit of enclosed area,*

$$\Omega = \frac{\Gamma}{A}. \text{ Thus},$$

$$\Omega = \frac{d\Gamma}{\Delta x . \Delta y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad ...(5.29)$$

If *a flow possesses vorticity, it is rotational.* **Rotation** ω *(omega) is defined as one-half of the vorticity*, or

$$\omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right]$$

The flow is irrotational if rotation ω *is zero.*

For a three-dimensional flow the rotation is possible about three axes. The expressions for rotation ω_z, ω_x *and* ω_y can be obtained in like manner:

$$\left.\begin{aligned} \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \end{aligned}\right\} \quad ...(5.30)$$

The motion is described as irrotational when the components of rotation or vorticity are 'zero'
throughout certain portion of the fluid. When torque is applied to the fluid particle it will give rise to rotation; the torque is due to shear stress. Therefore, the rotation of fluid particle will always be associated with shear stress. As
the shear stresses, in turn, depend upon the viscosity, the rotational flow occurs where the viscosity
effects are predominant. However, in the cases where the viscosity effects are small, the flow is
sometimes assumed to be irrotational.

In the vector notation, the above equation can be rewritten as:

$$\omega = \frac{1}{2}[\omega_x i + \omega_y j + \omega_z k]$$

$$= \frac{1}{2}(\Delta \times V)$$

The vector $(\Delta \times V)$ is the *curl* of velocity vector.

Vorticity in a fluid motion is taken numerically equal to *twice the value of rotation.*

$$\text{Vorticity, } \Omega = \text{curl } V = (\Delta \times V)$$

Which may be expressed as:

$$\Omega = (\nabla \times V)\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \Omega_x i + \Omega_y j + \Omega_z k$$

The vorticity components are separately given by:

$$\Omega_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\Omega_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\Omega_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$