- Determine the components of rotation for the following velocity field pertaining to the flow of an incompressible fluid: u = Cyz; v = Czx; w = Cxy, where C = constant. State whether the flow is rotational or irrotational

Solution. *Given:* $u = Cyz;\ v = Czx;\ w = Cxy$ ... Velocity field

The components of rotation are:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(Cx - Cx) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(Cy - Cy) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(Cz - Cz) = 0$$

- Since each of the rotation components is zero, the given flow field represents irrotational flow. (Ans.)

. Determine the components of rotation about the various axes for the following flows:

(i) $u = y^2,\ v = -3x$

(ii) $u = 3xy,\ v = \frac{3}{2}x^2 - \frac{3}{2}y^2$

(iii) $u = xy^3z,\ v = -y^2z^2,\ w = yz^2 - \frac{y^3z^2}{2}$

Solution. The components of rotation about the various axes are:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

(i) $u = y^2;\ v = -3x$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-3 - 2y) \text{ (Ans.)}$$

As the flow is two-dimensional in x–y plane, $\omega_x = \omega_y = 0$ (Ans.)

(ii) $u = 3xy,\ v = \frac{3}{2}x^2 - \frac{3}{2}y^2$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(3x - 3x) = 0 \text{ (Ans.)}$$

As the flow is two-dimensional in the $x - y$ plane, $\omega_x = \omega_y = 0$ (Ans.)

iii) $u = xy^3z;\ v = -y^2z^2;\ w = yz^2 - \frac{y^3z^2}{2}$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0 - 3xy^2z) = -\frac{3}{2}xy^2z \text{ (Ans.)}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(z^2 - \frac{3y^2z^2}{2} + 2y^2z\right) \text{ (Ans.)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(xy^3 - 0) = \frac{1}{2}xy^3 \text{ (Ans.)}$$

VELOCITY POTENTIAL AND STREAM FUNCTION

- Velocity Potential
- The velocity potential is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is denoted by ϕ (phi). Thus mathematically the velocity potential is defined as:
- ϕ = f (x, y, z, t) ...for unsteady flow, and, ϕ = f (x, y, z) ...for steady flow;

$$u = -\frac{\partial \phi}{\partial x} \quad v = -\frac{\partial \phi}{\partial y} \quad w = -\frac{\partial \phi}{\partial z}$$

- where, u, v and w are the components of velocity in the x, y and z directions respectively. The negative sign signifies that ϕ decreases with an increase in the values of x, y and z. In other words it indicates that the flow is always in the direction of decreasing ϕ. For an incompressible steady flow the continuity equation is given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- By substituting the values of u, v and w in terms of ϕ from eqn. , we get:

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

$$\frac{d^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

- This equation is known **as Laplace equation.**Thus any function ϕ that satisfies the Laplace equation will correspond to some case of fluid
- flow.

- The rotational components are given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- By substituting the values of u, v and w in term of ϕ from eqn. (5.35), we get:

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y\,\partial z} + \frac{\partial^2 \phi}{\partial z\,\partial y}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z\,\partial x} + \frac{\partial^2 \phi}{\partial x\,\partial z}\right]$$

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x\,\partial y} + \frac{\partial^2 \phi}{\partial y\,\partial x}\right]$$

- However, if ϕ is a continuous function then

$$\frac{\partial^2 \phi}{\partial y\,\partial z} = \frac{\partial^2 \phi}{\partial z\,\partial y};\ \frac{\partial^2 \phi}{\partial z\,\partial x} = \frac{\partial^2 \phi}{\partial x\,\partial z};\ \text{and}\ \frac{\partial^2 \phi}{\partial x\,\partial y} = \frac{\partial^2 \phi}{\partial y\,\partial x}$$

$$\omega_x = \omega_y = \omega_z = 0 \qquad \text{i.e. the flow is } \textit{irrotational}.$$

- Thus if velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady, incompressible, irrotational flow. Often an irrotational flow is known as potential flow.

Equipotential line

- Equipotential line:
- An equipotential line is one along which velocity potential ϕ is constant. i.e. For equipotential line, ϕ = constant.
- ∴ df = 0 But, ϕ = f(x, y) for steady flow.

$$\therefore \qquad d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$\text{But,} \qquad \frac{\partial \phi}{\partial x} = -u \text{ and } \frac{\partial \phi}{\partial y} = -v$$

$$\therefore \qquad d\phi = -udx - vdy = -(udx + vdy)$$

For equipotential line, $d\phi = 0$

$$\text{or,} \qquad -(udx + vdy) = 0$$

$$\text{or,} \qquad (udx + vdy) = 0$$

$$\text{or,} \qquad \frac{dy}{dx} = -\frac{u}{v}$$

where, $\frac{dy}{dx}$ = slope of equipotential line.

Stream Function

- The stream function is defined as a function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to this direction. It is denoted by ψ (psi)
- In case of two-dimensional flow, the stream function may be defined mathematically as
- ψ = f (x, y, t) ...for unsteady flow, and
- ψ = f (x, y) ... for steady flow

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

For two-dimensional flow the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the values of u and v from eqn. (5.38), we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Hence, existence of ψ means a possible case of fluid flow.

— The flow may be 'rotational' or 'irrotational'.

The rotational component ω_z is given by:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right]$$

$$\omega_z = -\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

This equation is known as ***Poisson's equation.***
For an *irrotational flow*, since ωz = 0, eqn. becomes:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad i.e., \quad \Delta^2 \psi = 0$$

which is the *Laplace equation in* ψ.

In the *polar co-ordinates*:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; v_\theta = \frac{\partial \psi}{\partial r}$$

Properties of stream function

The properties of stream function are:

1. On any stream line, ψ is constant everywhere.
ψ = constant, represents the family of stream lines.
ψ = constant, is a stream line equation.
2. If the flow is continuous, the flow around any path in the fluid is zero.
3. The rate of change of ψ with distance in arbitrary direction is proportional to the component of velocity normal to that direction.
4. The algebraic sum of stream function for two incompressible flow patterns is the stream function for the flow resulting from the superimposition of these patterns.