Cauchy Riemann equations:

- These equations, in hydrodynamics, are sometimes called "Cauchy Riemann equations".

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= -\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned}\right\}$$

- **Relation between Stream Function and Velocity Potential**
- a stream function is that the difference of its values at two points
- represent the flow across any line joining the points. Thus if two lie on the same stream line, then,
- there being no flow across a stream line, the difference between the stream functions ψ1 and ψ2 = 0;
- this means the streamline is given by:
- ψ = constant.
- Similarly, φ = constant, represents a case for which the velocity potential is same at every point, and hence it represents an equipotential line.
- Let, two curves φ = constant and ψ = constant intersect each other at any point. At the point of
- intersection the slopes are:

$$\textit{For the curve } \phi = \textit{constant:} \quad \text{Slope} = \frac{\partial y}{\partial x} = \frac{\left(\frac{\partial\phi}{\partial x}\right)}{\left(\frac{\partial\phi}{\partial y}\right)} = \frac{-u}{-v} = \frac{u}{v}$$

$$\textit{For the curve } \psi = \textit{constant:} \quad \text{Slope} = \frac{\partial y}{\partial x} = \frac{\left(\frac{\partial\psi}{\partial x}\right)}{\left(\frac{\partial\psi}{\partial y}\right)} = \frac{-v}{+u} = -\frac{v}{u}$$

Now, product of the slopes of these curves

$$= \frac{u}{v} \times -\frac{v}{u} = -1$$

- It shows that these two sets of curves, viz stream lines and equipotential lines intersect each other orthogonally at all points of intersection.

- Verify whether the following functions are valid potential functions:
- (i) ϕ = A(x2 – y2) (ii) ϕ = A cos x

Solution. (*i*) $\phi = A\,(x^2 - y^2)$:

$$\frac{\partial \phi}{\partial x} = 2\,Ax\,; \quad \frac{\partial \phi}{\partial y} = -2\,Ay$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\,A\,; \quad \frac{\partial^2 \phi}{\partial y^2} = -2\,A$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2A + (-2A) = 0$$

Hence, $\phi = A\,(x^2 - y^2)$ **is a valid potential function (Ans.)**

(*ii*) $\phi = A \cos x$:

$$\frac{\partial \phi}{\partial x} = -A \sin x\,; \quad \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = -A \cos x\,; \quad \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -A \cos x \neq 0$$

Hence, $\phi = A \cos x$ **is not a valid function (Ans.)**

- Which of the following functions represent possible irrotational flow.?

(i) $\psi = A(x^2 - y^2)$ (ii) $\psi = xy$

(iii) $\phi = \left(r - \frac{2}{r}\right)\sin\theta$ (iv) $\phi = Ur\cos\theta + \frac{U}{r}\cos\theta$

Solution. *For an irrotational fluid flow phenomenon ϕ as well ψ satisfy Laplace equation.*

(i) $\psi = A(x^2 - y^2)$:

$$\frac{\partial\psi}{\partial x} = 2Ax; \quad \frac{\partial\psi}{\partial y} = -2Ay$$

$$\frac{\partial^2\psi}{\partial x^2} = 2A; \quad \frac{\partial^2\psi}{\partial y^2} = -2A$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 2A - 2A = 0$$

Hence, $\psi = A(x^2 - y^2)$ represents a **possible irrotational flow** (Ans.)

(ii) $\psi = xy$:

$$\frac{\partial\psi}{\partial x} = y; \quad \frac{\partial\psi}{\partial y} = x$$

$$\frac{\partial^2\psi}{\partial x^2} = 0; \quad \frac{\partial\psi}{\partial y} = 0$$

$$\therefore \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Hence, $\psi = xy$ represents a **possible irrotational flow.** (Ans.)

(*iii*) $\phi = \left(r - \frac{2}{r}\right)\sin\theta$:

Laplace equation in radial coordinates (r, θ) is given as:

$$\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$\frac{\partial\phi}{\partial r} = \left(1 + \frac{2}{r^2}\right)\sin\theta$$

$$\frac{\partial^2\phi}{\partial r^2} = -\frac{2}{r^3}\sin\theta$$

$$\frac{\partial\phi}{\partial\theta} = \left(r - \frac{2}{r}\right)\cos\theta$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -\left(r - \frac{2}{r}\right)\sin\theta = \left(\frac{2}{r} - r\right)\sin\theta$$

L.H.S. of Laplace equation is:

$$\frac{1}{r} \times \left(1 + \frac{2}{r^2}\right)\sin\theta - \frac{2}{r^3}\sin\theta + \frac{1}{r^2}\left(\frac{2}{r} - r\right)\sin\theta$$

$$= \sin\theta\left(\frac{1}{r} + \frac{2}{r^3} - \frac{2}{r^3} + \frac{2}{r^3} - \frac{1}{r}\right)$$

$$= \sin\theta\left(\frac{2}{r^3}\right) \neq 0$$

Hence, the given function doesnot represent any possible irrotational flow. (Ans.)

- The velocity components in a fluid flow are given by: u = 2xy; v = a2 + x2 – y2
- (i) Show that the flow is possible. (ii) Derive the relative stream function

Solution. *Given:* $u = 2xy;\ v = a^2 + x^2 - y^2$ ...Velocity components

(*i*)
$$\frac{\partial u}{\partial x} = 2y\ ;\ \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2y + (-2y) = 0$$

The *continuity equation for steady, incompressible flow is satisfied.*

Hence, **flow is possible. (Ans.)**

(*ii*) The stream function ψ is related to *u* and *v* as:

$$u = \frac{\partial \psi}{\partial y} = 2xy$$

or,
$$\psi = \int 2xy\,dy = xy^2 + f(x) \qquad \text{...}(i)$$

$$-\frac{\partial \psi}{\partial x} = -y^2 - f'(x) = v = a^2 + x^2 - y^2$$

Hence,
$$f'(x) = -(a^2 + x^2)$$

or,
$$f(x) = -a^2x - \frac{x^3}{3} + \text{constant}$$

Inserting for $f(x)$ in eqn. (*i*) we get:

$$\psi = xy^2 - ax^2 - \frac{x^3}{3} + \text{constant}$$

Thus, the relative $\psi = xy^2 - a^2x - \dfrac{x^3}{3}$ **+ constant (Ans.)**

- A flow is described by the stream function ψ = 4xy . Locate the point at which the velocity vector has a magnitude 7 units and makes an angle of 150° with X-axis.

Solution. Stream function, $\psi = 4xy$ ...Given

The velocity components for the given flow field are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(4xy) = 4x$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(4xy) = -4y$$

$$V = \sqrt{u^2 + v^2} \quad \text{or} \quad 7 = \sqrt{(4x)^2 + (-4y)^2} = 4\sqrt{x^2 + y^2} \quad \text{...}(i)$$

Also, $\tan\theta = \frac{v}{u}$ or $\tan 150° = \frac{-4y}{4x} = -\frac{y}{x}$

or, $-0.577 = -\frac{y}{x}$ or $y = 0.577\,x$

Substituting for y in eqn. (i), we get:

$$7 = 4\sqrt{x^2 + (0.577\,x)^2} = 4.62\,x$$

$$\therefore \quad x = \frac{7}{4.62} = \mathbf{1.515\ (Ans.)}$$