

# Fluid Properties

- Similarly, work done by the pressure at MM in moving the liquid to M'M' =  $-p_2.A_2 . dl_2$
- (– ve sign indicates that direction of  $p_2$  is opposite to that of  $p_1$ )
- $\therefore$  Total work done by the pressure
- =  $p_1 . A_1 dl_1 - p_2 A_2 dl_2$
- =  $p_1 . A_1 dl_1 - p_2 A_1 dl_1$  ( $\because A_1 dl_1 = A_2 dl_2$ )
- =  $A_1 . dl_1 (p_1 - p_2)$

$$= \frac{W}{w} (p_1 - p_2) \quad \left( \because A_1 . dl_1 = \frac{W}{w} \right)$$

$$\text{Loss of potential energy} = W(z_1 - z_2)$$

$$\text{Gain in kinetic energy} = W \left( \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \right) = \frac{W}{2g} (V_2^2 - V_1^2)$$

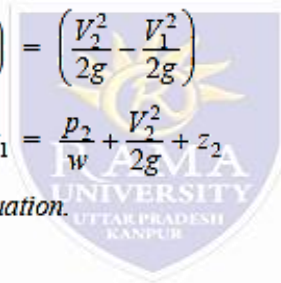
Also, Loss of potential energy + work done by pressure = Gain in kinetic energy

$$\therefore W(z_1 - z_2) + \frac{W}{w} (p_1 - p_2) = \frac{W}{2g} (V_2^2 - V_1^2)$$

$$\text{or, } (z_1 - z_2) + \left( \frac{p_1}{w} - \frac{p_2}{w} \right) = \left( \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \right)$$

$$\text{or, } \frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

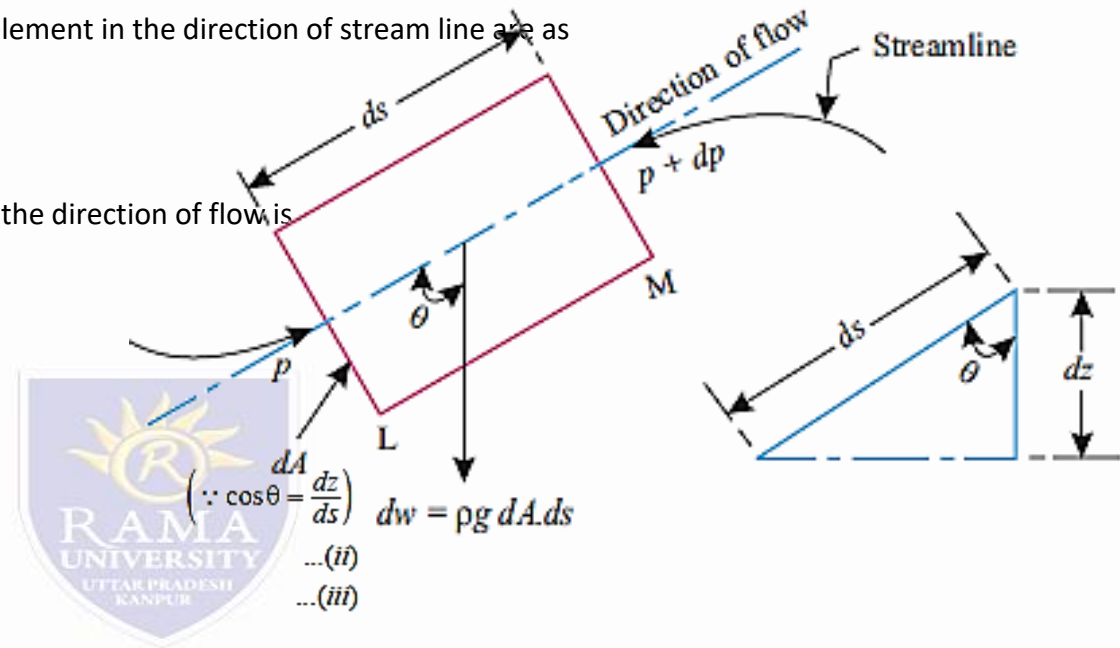
which proves *Bernoulli's equation*.



- which proves Bernoulli's equation.
- Assumptions:
- It may be mentioned that the following assumptions are made in the derivation of Bernoulli's equation:
- 1. The liquid is ideal and incompressible.
- 2. The flow is steady and continuous.
- 3. The flow is along the stream line, i.e., it is one-dimensional.
- 4. The velocity is uniform over the section and is equal to the mean velocity.
- 5. The only forces acting on the fluid are the gravity forces and the pressure forces.

# EULER'S EQUATION FOR MOTION

- Consider steady flow of an ideal fluid along the stream tube. Separate out a small element of fluid of cross-sectional area  $dA$  and length  $ds$  from stream tube as a free body from the moving fluid. Fig. shows such a small element LM of fluid of cross-section area  $dA$  and length  $ds$ .
- Let,  $p$  = Pressure on the element at L,  $p + dp$  = Pressure on the element at M, and  $V$  = Velocity of the fluid element.
- The external forces tending to accelerate the fluid element in the direction of stream line are as follows:
- 1. Net pressure force in the direction of flow is,
- $p \cdot dA - (p + dp) \cdot dA = -dp \cdot dA$  ... (i)
- 2. Component of the weight of the fluid element in the direction of flow is
- $= -\rho \cdot g \cdot dA \cdot ds \cdot \cos\theta$



$$= -\rho g \cdot dA \cdot ds \left( \frac{dz}{ds} \right)$$

$$= -\rho g \cdot dA \cdot dz$$

$$\text{Mass of the fluid element} = \rho \cdot dA \cdot ds$$

The acceleration of the fluid element

$$a = \frac{dV}{dt} = \frac{dV}{ds} \times \frac{ds}{dt} = V \cdot \frac{dV}{ds} \quad \dots (iv)$$

∴, according to Newton's second law of motion, Force = Mass × acceleration

$$\therefore -dp \cdot dA - \rho \cdot g \cdot dA \cdot dz = p \cdot dA \cdot ds \times V \cdot \frac{dV}{ds}$$

Dividing both sides by  $\rho \cdot dA$ , we get:

$$\frac{-dp}{\rho} - g \cdot dz = V \cdot dV$$

$$\text{or, } \frac{dp}{\rho} + V \cdot dV + g \cdot dz = 0 \quad \dots (6.3)$$

This is the required **Euler's equation** for motion, and is in the form of *differential equation*.

Integrating the above eqn., we get:

$$\frac{1}{\rho} \int dp + \int V.dV + \int g.dz = \text{constant}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Dividing by  $g$ , we get:

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$

or, 
$$\frac{p}{w} + \frac{V^2}{2g} + z = \text{constant}$$

or, in other words,

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

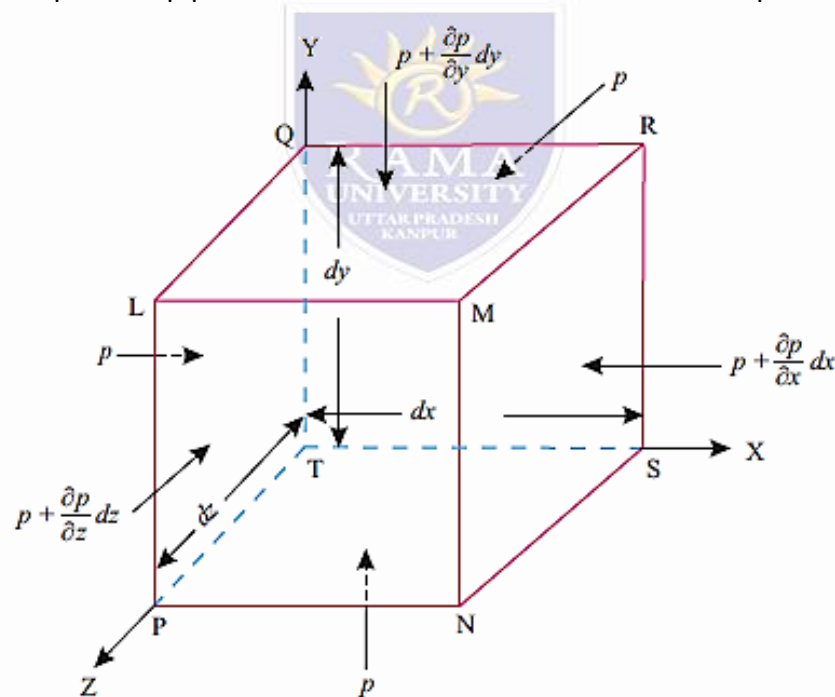
which proves *Bernoulli's equation*.

# Fluid Properties

- Euler's equation in Cartesian coordinates:
- Consider an infinitely small mass of fluid enclosed in an elementary parallelepiped of sides  $dx$ ,  $dy$  and  $dz$  as shown in Fig. 6.3. The motion of the fluid element is influenced by the following forces:
- (i) Normal forces due to pressure: The intensities of hydrostatic pressure acting normal to each face of the parallelepiped are shown in Fig. **The net pressure force in the X-direction**

$$= p \cdot dy \cdot dz - \left( p + \frac{\partial p}{\partial x} dx \right) dy \cdot dz$$

- ii) Gravity or body force:
- Let  $B$  be the body force per unit mass of fluid having components  $B_x$ ,  $B_y$  and  $B_z$  in the X, Y and Z directions respectively.
- Then, the body force acting on the parallelepiped in the direction of X-coordinate is  $= B_x \cdot p \cdot dx \cdot dy \cdot dz$ .



# Fluid Properties

## (iii) Inertia forces:

The inertia force acting on the fluid mass, along the  $X$ -coordinate is given by,

$$\text{Mass} \times \text{acceleration} = \rho \cdot dx \cdot dy \cdot dz \cdot \frac{du}{dt}$$

As per Newton's second law of motion summation of forces acting in the fluid element in any direction equals the resulting inertia forces in that direction. Thus, along  $X$ -direction:

$$B_x \cdot \rho \cdot dx \cdot dy \cdot dz - \frac{\partial p}{\partial x} dx \cdot dy \cdot dz = \rho \cdot dx \cdot dy \cdot dz \cdot \frac{du}{dt}$$

Dividing both sides by  $\rho \cdot dx \cdot dy \cdot dz$ , we have:

$$B_x - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{du}{dt} \quad \dots(i)$$

In this equation each term has dimensions of force per unit mass or acceleration. Obviously the total acceleration in a given direction is prescribed by the algebraic sum of the body force and the pressure gradient in that direction since the velocity components are functions of position and time, i.e.,  $u = f(x, y, z, t)$ , therefore, the total derivative of velocity  $u$  in the  $X$ -direction can be written as:

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{dx}{dt} = u, \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w, \text{ we have:}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

Combining eqns. (i) and (ii), we get the force components as:

$$B_x - \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

Similarly,

$$B_y - \frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$$

and,

$$B_z - \frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

For steady flow:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$$

Thus, the Euler's equation for a steady three-dimensional flow can be written as:

$$B_x - \frac{1}{\rho} \frac{\partial p}{\partial x} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$B_y - \frac{1}{\rho} \frac{\partial p}{\partial y} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$$

$$B_z - \frac{1}{\rho} \frac{\partial p}{\partial z} = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

# Fluid Properties

- In Euler's equation each term represents force per unit mass. Thus, if each equation is multiplied by the respective projections of the elementary displacement, the resulting equation would represent energy. Thus, in order to get total energy in the three-dimensional-steady-incompressible flow, the energy terms can be combined as follows:

$$B_x dx - \frac{1}{\rho} \frac{\partial p}{\partial x} dx = u \frac{\partial u}{\partial x} dx + v \frac{\partial u}{\partial y} dx + w \frac{\partial u}{\partial z} dx$$

$$B_y dy - \frac{1}{\rho} \frac{\partial p}{\partial y} dy = u \frac{\partial v}{\partial x} dy + v \frac{\partial v}{\partial y} dy + w \frac{\partial v}{\partial z} dy$$

$$B_z dz - \frac{1}{\rho} \frac{\partial p}{\partial z} dz = u \frac{\partial w}{\partial x} dz + v \frac{\partial w}{\partial y} dz + w \frac{\partial w}{\partial z} dz$$

where,  $V$  = Total velocity vector.

When gravity is the only body force acting on the third element, then:

$$B_x = 0, B_z = 0 \text{ and } B_y = -g$$

$B_y = -g$  since the gravitational force acts in the downward direction which is negative 'with' respect to  $Y$ , which is positive upward. Inserting these values in (xviii), we get:

$$-g - \frac{1}{\rho} dp = \frac{1}{2} d(V^2)$$

$$-g - \frac{1}{\rho} dp = VdV$$

$$\frac{dp}{\rho} + VdV + g = 0 \text{ which is the same as Euler's}$$

From the equation of a stream line in a three-dimensional flow, we have:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$u dy = v dx; v dz = w dy; u dz = w dx$$

Substituting these values in eqns. (ix), (x) and (xi), we get:

$$B_x dx - \frac{1}{\rho} \frac{\partial p}{\partial x} dx = u \frac{\partial u}{\partial x} dx + u \frac{\partial u}{\partial y} dy + u \frac{\partial u}{\partial z} dz \quad \dots(xiii)$$

$$B_y dy - \frac{1}{\rho} \frac{\partial p}{\partial y} dy = v \frac{\partial v}{\partial x} dx + v \frac{\partial v}{\partial y} dy + v \frac{\partial v}{\partial z} dz \quad \dots(xiii)$$

$$B_z dz - \frac{1}{\rho} \frac{\partial p}{\partial z} dz = w \frac{\partial w}{\partial x} dx + w \frac{\partial w}{\partial y} dy + w \frac{\partial w}{\partial z} dz \quad \dots(xiv)$$

Acceleration terms are of form  $u \frac{\partial u}{\partial x}$  which can be replaced by  $\frac{1}{2} \frac{\partial(u^2)}{\partial x}$ . Thus,

$$B_x dx - \frac{1}{\rho} \frac{\partial p}{\partial x} dx = \frac{1}{2} \left[ \frac{\partial}{\partial x} (u^2) dx + \frac{\partial}{\partial y} (u^2) dy + \frac{\partial}{\partial z} (u^2) dz \right] = \frac{1}{2} d(u^2) \quad \dots(xv)$$

Similarly,  $B_y dy - \frac{1}{\rho} \frac{\partial p}{\partial y} dy = \frac{1}{2} d(v^2) \quad \dots(xvi)$

and,  $B_z dz - \frac{1}{\rho} \frac{\partial p}{\partial z} dz = \frac{1}{2} d(w^2) \quad \dots(xvii)$