Free Vortex Flow

- Free vortex flow is one in which the fluid mass rotates without any external impressed contact force. The whole fluid mass rotates either due to fluid pressure itself or the gravity or due to rotation previously imparted. The free vortex motion is also called potential vortex or irrotational vortex.
- Example:
- 1. Flow around a circular bend.
- 2. A whirlpool in a river.
- 3. Flow of liquid in a centrifugal pump casing after it has left the impeller.
- 4. Flow of water in a turbine casing before it enters the guide vanes.
- 5. Flow of liquid through a hole/outlet provided at the bottom of a shallow vessel (e.g., wash basin, bath tub, etc.)
- In free vortex the relation between velocity and radius is obtained by putting the value of external torque equal to zero, or, the time rate change of angular momentum (i.e., moment of momentum) must be zero.
- Let us consider a particle of mass m at a radius distance r from the axis of rotation, having a tangential velocity, v. Then:
- Moment of momentum = (m × v) × r = mvr
- ∴ Time rate of change of momentum

$$= \frac{\partial}{\partial t}(mvr)$$

But for the vortex, $= \frac{\partial}{\partial t}(mvr) = 0$

Integrating, we get: $mvr = \text{constant}$

Since m is constant, $vr = \text{constant} = C$

where C is a constant and is known as *strength of vortex*.

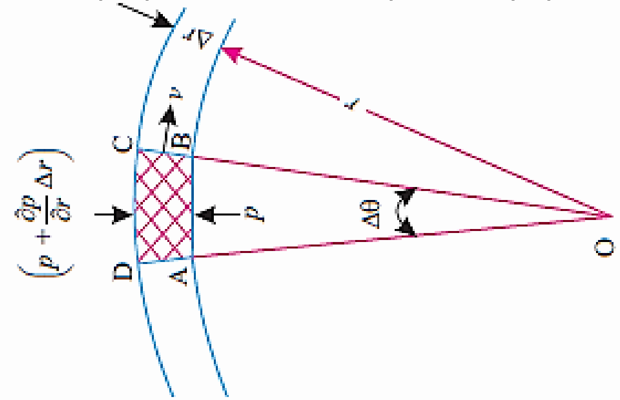
$$\therefore \quad v = \frac{C}{r}$$

or, $$v \propto \frac{1}{r}$$

i.e. tangentital velocity is *inversely proportional to distance r*.

Equation of Motion for Vortex Flow.

- ABCD is fluid element rotating at a uniform velocity in a horizontal plane about an axis perpendicular to the plane of paper and passing through 0.
- Let, r = Radius of the element from O,
- Δr = Radial thickness of the element,
- ΔA = Area of cross-section of element, and
- Δθ = Angle subtended by the element at O.
- The various forces acting on the element are:
- 1. Centrifugal force mv2/r , acting away from the centre, O,
- 2. Pressure force pΔA on the face AB, and



3. Pressure force $\left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A$ on the face *CD*.

Equating the forces in the radial direction, we get:

$$\left(p + \frac{\partial p}{\partial r}\,.\,\Delta r\right)\Delta A - p\Delta A = \frac{mv^2}{r}$$

But, m = mass density × volume = ρ × Δ*A* × Δ*r*

$$\therefore \quad \left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A - p\Delta A = \rho\Delta A\,\Delta r\frac{v^2}{r}$$

$$\text{or,} \quad \rho\Delta A + \frac{\partial p}{\partial r}\Delta r\,\Delta A - p\Delta A = \rho\Delta A\Delta r\frac{v^2}{r}$$

$$\text{or,} \quad \frac{\partial p}{\partial r}\Delta r\,\Delta A = \rho\Delta A\Delta r\frac{v^2}{r}$$

$$\text{or,} \quad \frac{\partial p}{\partial r} = \frac{\rho v^2}{r}$$

The expression $\frac{\partial p}{\partial r}$ is called *pressure gradient* in the radial direction.

$\left(\text{Since } \frac{\partial p}{\partial r} \text{ is } +ve, \text{ therefore, pressure increases with the increase of radius } r.\right)$

In the vertical plane, the variation of pressure is given by the hydrostatic law, *i.e.*,

$$\frac{\partial p}{\partial z} = -\rho g$$

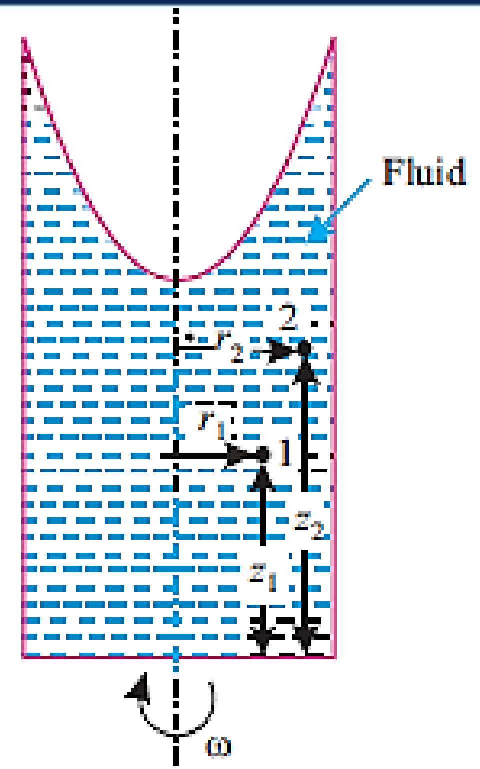
As the pressure is a function of *r* and *z*, therefore total derivative of *p*,

$$\partial p = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$$

Substituting the values of $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial z}$ from eqns. (6.35) and (6.36) respectively, we g

$$dp = \frac{\rho v^2}{r}dr - \rho g\,dz$$

Equation of Forced Vortex Flow



- In case of forced vortex flow,
- v = ωr ...(where, ω = constant angular velocity)
- Putting the value of v in eqn .pressure variation , we get

$$dp = \frac{\rho\omega^2 r^2}{r} dr - \rho g\, dz$$

$$dp = \rho\omega^2 r dr - \rho g dz$$

Considering points 1 and 2 in the fluid having forced vortex flow (Fig. 6.66) and integrating the above eqn. for these points, we get

$$\int_1^2 dp = \int_1^2 \rho\omega^2 r\, dr - \int_1^2 \rho g\, dz$$

$$[p]_1^2 = \rho\omega^2 \left[\frac{r^2}{2}\right]_1^2 - \rho g [z]_1^2$$

$$(p_2 - p_1) = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$$

$$= \frac{\rho}{2}(\omega^2 r_2^2 - \omega^2 r_1^2) - \rho g(z_2 - z_1)$$

$$= \frac{\rho}{2}(v_2^2 - v_1^2) - \rho g(z_2 - z_1)$$

$[\because v_1 = \omega_1 r_1 \text{ and } v_2 = \omega_2 r_2]$

When the points 1 and 2 lie on the free surface of the liquid, then p1 = p2 and the above equation becomes:

$$0 = \frac{\rho}{2}(v_2^2 - v_1^2) - \rho g(z_2 - z_1)$$

$$g(z_2 - z_1) = \left(\frac{v_2^2 - v_1^2}{2}\right)$$

$$z_2 - z_1 = \frac{v_2^2 - v_1^2}{2g}$$

— When the point 1 lies on the axis of rotation, then: v1 = ωr1 = ω × 0 = 0; the above eqn. reduces to:

$$z_2 - z_1 = \frac{v_2^2}{2g}$$

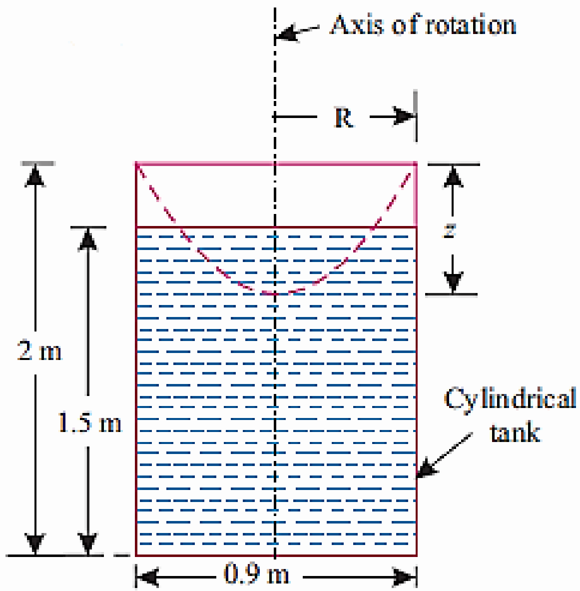
$z_2 - z_1 = z$ (say), then we have:

$$z = \frac{v_2^2}{2g} = \frac{\omega^2 r_2^2}{2g}$$

Thus, *z* varies with square of *r*. Hence eqn. (6.38) is an equation of *parabola* which means that the free surface of the liquid is a paraboloid.

- A cylindrical tank 0.9 m in diameter and 2 m high open at top is filled with water to a depth of 1.5 m. It is rotated about its vertical axis at N. r.p.m. Determine the value of N whichwill raise water level even with the brim (GATE)

- Solution. Refer to Fig. 6.68. Given: Radius, R = 0.9/ =2 0.45m
- Length, = 2m; Initial height of
- water = 1.5 m.
- Speed which will raise water level even with brim, N:
- When the vessel is rotated, parboloid is formed.
- Volume of air bef
- ore rotation = Volume of air after rotation



$$\pi R^2 \times 2 - \pi R^2 \times 1.5 = \frac{1}{2}\pi R^2 z$$

or, $z = 1.0$ m

Using the relation:

$$z = \frac{\omega^2 r^2}{2g}, \text{ we get:}$$

$$1.0 = \frac{\omega^2 R^2}{2 \times 9.81} \quad (\text{Here, } r = R)$$

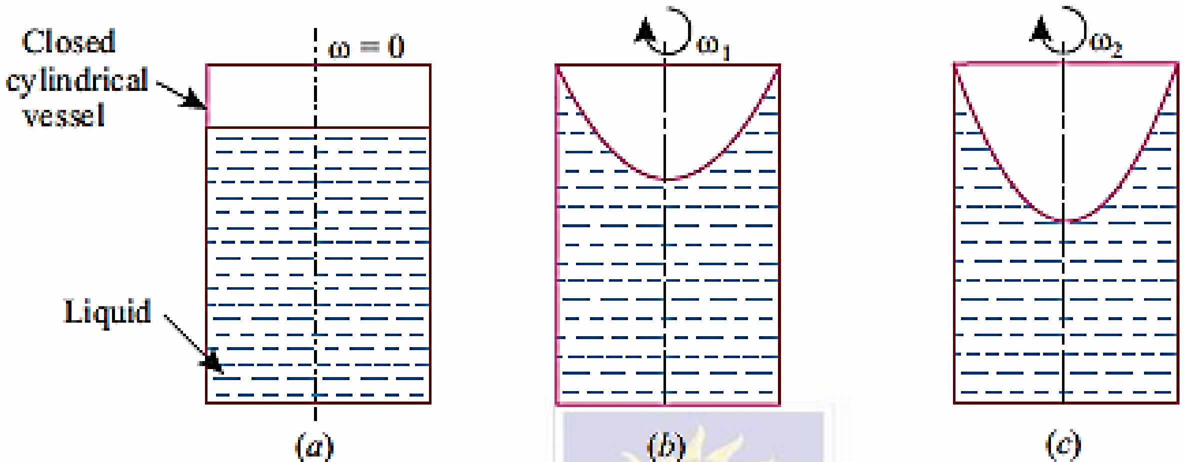
$$\omega = \sqrt{\frac{1.0 \times 2 \times 9.81}{(0.45)^2}} = 9.843$$

But, $$\omega = \frac{2\pi N}{60}$$

$$\therefore \quad N = \frac{9.843 \times 60}{2\pi} = 93.99 \text{ r.p.m. (Ans.)}$$

Rotation of Liquid in a Closed Cylindrical Vessel

- When a cylindrical vessel sealed at the top and filled with some liquid is rotated about its vertical geometrical axis, the shape of paraboloid formed due to rotation of the vessel will be as shown in Fig 6.72 for different speeds of rotation.



- — Fig. 6.72 (a) shows the cylindrical vessel when it is stationary (i.e., it is not rotated, ω = 0)
- — Fig. 6.72 (b) shows the shape of the paraboloid formed when the speed of rotation is ω1.
- — Fig. 6.72 (c) shows the shape of the paraboloid formed when the speed of rotation is
- ω 2(ω 2 > ω 1). In this case the following are unknown:
- 1. Radius of the parabola at the top of the vessel, and
- 2. Height of the parabola formed corresponding to the angular speed, ω2.
- To solve these, two unknown equations are required:
- (i) One equation is:

$$z = \frac{\omega^2 r^2}{2g}$$

- (ii) Second equation is from the fact that for closed Vessel:
- Volume of air before rotation = Volume of air after rotation
- Volume of air before rotation = Volume of closed vessel – volume of liquid in the vessel
- Volume of air after rotation = Volume of paraboloid formed

$$= \frac{1}{2}\pi r^2 \times z .$$

Equation of Free Vortex Flow

In the case of free vortex flow, from eqn. (6.34), we have:

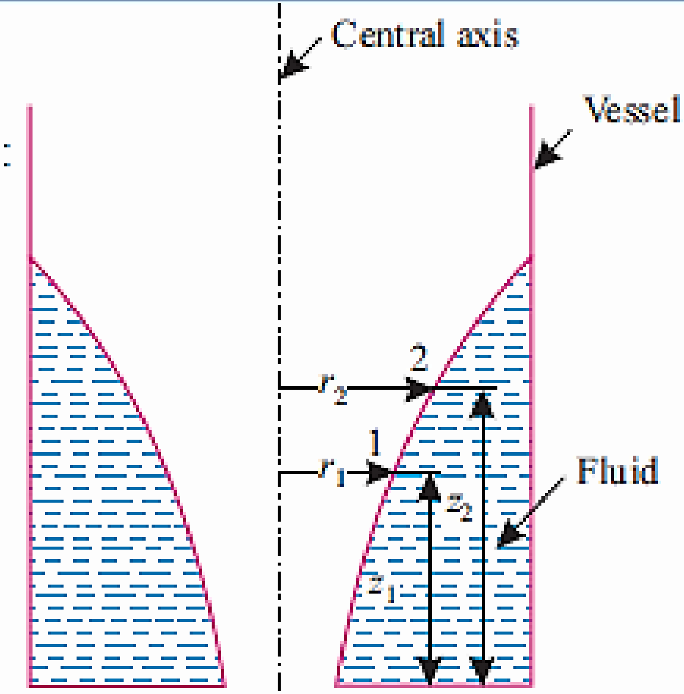
$$v = \frac{C}{r}$$

Substituting the value of v in eqn. (6.37), we get:

$$dp = \frac{\rho v^2}{r} dr - \rho g\, dz$$

$$= \rho \times \frac{C^2}{r^2 \times r} dr - \rho g dz$$

$$= \frac{\rho C^2}{r^3} dr - \rho g\, dz$$



Refer to Fig. 6.79. Consider two points 1and 2 in the fluid having radii r1 and r2 respectively from the central axis, their heights being z1 and z2 from bottom of the vessel. Integrating the above equation for the points 1 and 2, we get:

$$\int_1^2 dp = \int_1^2 \frac{\rho C^2}{r^3} dr - \int_1^2 \rho g\, dz$$

$$p_2 - p_1 = \rho C^2 \int_1^2 \frac{dr}{r^3} - \rho g \int_1^2 dz$$

$$= \rho C^2 \left[\frac{r^{-3+1}}{-3+1}\right]_1^2 - \rho g (z_2 - z_1)$$

$$= \rho C^2 \left[-\frac{1}{2r^2}\right]_1^2 - \rho g (z_2 - z_1)$$

$$= -\frac{\rho C^2}{2}\left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] - \rho g (z_2 - z_1)$$

$$= -\frac{\rho}{2}[v_2^2 - v_1^2] - \rho g (z_2 - z_1) \quad \left(\because v_2 = \frac{C}{r_2}, v_1 = \frac{C}{r_1}\right)$$

$$= \frac{\rho}{2}(v_1^2 - v_2^2) - \rho g (z_2 - z_1)$$

Dividing both sides by ρg, we get:

$$\frac{p_2 - p_1}{\rho g} = \frac{v_1^2 - v_2^2}{2g} - (z_2 - z_1)$$

or,

$$\left(\frac{p_2}{\rho g} - \frac{p_1}{\rho g}\right) = \left(\frac{v_1^2}{2g} - \frac{v_2^2}{2g}\right) + (z_1 - z_2)$$

or,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$