Figure 4.30(b) shows the stresses acting on an oblique plane. The normal to the plane makes an angle $\theta$ with the $X$-axis. σ and $\tau$ are normal and shear stresses associated with this plane. Considering equilibrium of forces, it can be proved that,[2, 3]

$$\sigma = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{4.28}$$

and

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{4.29}$$

Differentiating Eq. (4.28) with respect to θ and setting the result to zero, we have

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{4.30}$$

Equation (4.30) defines two values of ($2\theta$), one giving the maximum value of normal stress and other the minimum value. If $\sigma_1$ and $\sigma_2$ are the maximum and minimum values of normal stress, then substituting Eq. (4.30) in Eq. (4.28), we get

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{4.31}$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{4.32}$$

$\sigma_1$ and $\sigma_2$ are called the principal stresses.

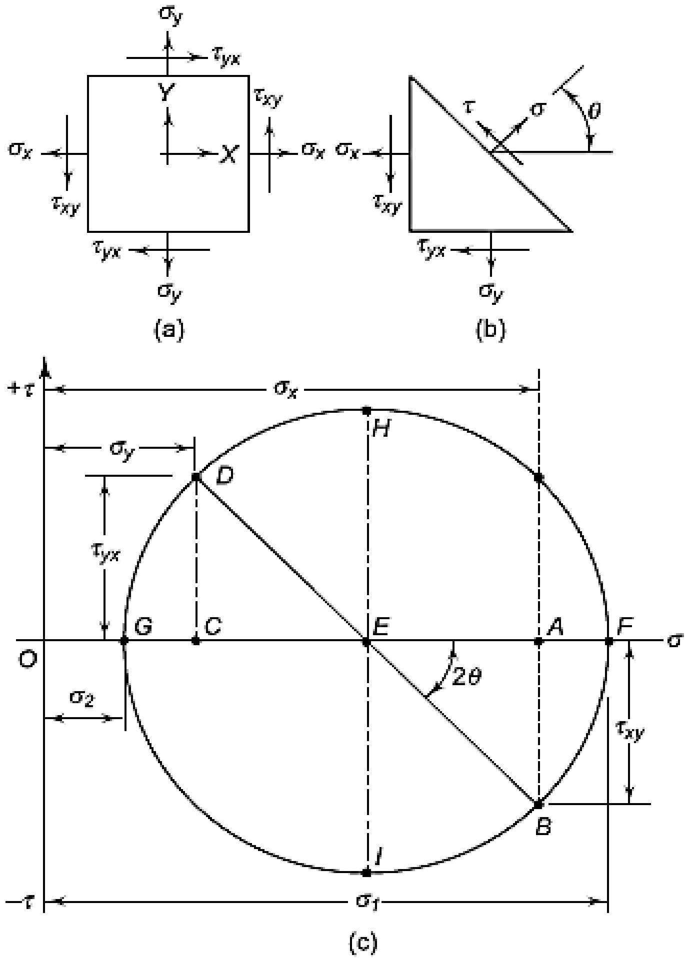


**Fig. 4.30** *(a) Two-Dimensional State of Stress (b) Stresses on Oblique Plane (c) Mohr's Circle Diagram*

Similarly, Eq. (4.29) is differentiated with respect to $\theta$ and the result is equated to zero. This gives the following condition:

$$\tan 2\theta = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) \tag{4.33}$$

Substituting Eq. (4.33) in Eq. (4.29),

$$\tau_{max.} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \tag{4.34}$$

$\tau_{max.}$ is called the principal shear stress.

One of the most effective methods to determine the principal stresses and the principal shear stress is the construction of *Mohr's circle diagram* as shown in Fig. 4.30 (c). It is a graphical method for the representation of stresses. The following conventions are used to construct the *Mohr's circle*:

(i) The normal stresses $\sigma_x$, $\sigma_y$, and the principal stresses $\sigma_1$, $\sigma_2$ are plotted on the abscissa. The tensile stress, considered as positive, is plotted to the right of the origin and the compressive stress, considered as negative, to its left.

(ii) The shear stresses $\tau_{xy}$, $\tau_{yx}$ and the principal shear stress $\tau_{max}$ are plotted on the ordinate. A pair of shear stresses is considered as positive if they tend to rotate the element clockwise, and negative if they tend to rotate it anticlockwise.

The Mohr's circle in Fig. 4.30 (c) is constructed by the following method:

(i) Plot the following points:

$\overline{OA} = \sigma_x$ $\qquad$ $\overline{OC} = \sigma_y$

$\overline{AB} = \tau_{xy}$ $\qquad$ $\overline{CD} = \tau_{yx}$

(ii) Join $\overline{DB}$. The point of intersection of $\overline{DB}$ and $\overline{OA}$ is $E$.

(iii) Construct Mohr's circle with $E$ as centre and $\overline{DB}$ as the diameter.

It can be proved that points $F$ and $G$ represent the maximum and minimum principal stresses $\sigma_1$ and $\sigma_2$ respectively. The two principal shear stresses $\pm\tau_{max}$ are denoted by points $H$ and $I$ respectively.

- THEORIES OF ELASTIC FAILURE
- Theories of failure discussed in this article are applicable to elastic failure of machine parts.
- Theories of elastic failure provide a relationship between the strength of machine component subjected to complex state of stresses with the mechanical properties obtained in tension test. With the help of these theories, the data obtained in the tension test can be used to determine the dimensions of the component, irrespective of the nature of stresses induced in the component due to complex loads. Several theories have been proposed, each assuming a different hypothesis of failure.
- The principal theories of elastic failure are as follows:
- (i) Maximum principal stress theory (Rankine's theory)
- (ii) Maximum shear stress theory (Coulomb, Tresca and Guest's theory)
- (iii) Distortion energy theory (Huber von Mises and Hencky's theory)
- (iv) Maximum strain theory (St. Venant's theory)
- (v) Maximum total strain energy theory (Haigh's theory)

- 1. MAXIMUM PRINCIPAL STRESS THEORY
- This criterion of failure is accredited to the British engineer WJM Rankine (1850). The theory states that the failure of the mechanical component subjected to bi-axial or tri-axial stresses occurs when the maximum principal stress reaches the yield or ultimate strength of the material.

If $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the three principal stresses at a point on the component and

$$\sigma_1 > \sigma_2 > \sigma_3$$

then according to this theory, the failure occurs whenever

$$\sigma_1 = S_{yt} \quad \text{or} \quad \sigma_1 = S_{ut} \qquad (4.35)$$

whichever is applicable.

The theory considers only the maximum of principal stresses and disregards the influence of the other principal stresses. The dimensions of the component are determined by using a factor of safety.

For tensile stresses,

$$\sigma_1 = \frac{S_{yt}}{(fs)} \quad \text{or} \quad \sigma_1 = \frac{S_{ut}}{(fs)} \qquad (4.36)$$

For compressive stresses,

$$\sigma_1 = \frac{S_{yc}}{(fs)} \quad \text{or} \quad \sigma_1 = \frac{S_{uc}}{(fs)} \qquad (4.37)$$

**Region of Safety**

The construction of region of safety for bi-axial stresses is illustrated in Fig. 4.31. The two principal stresses $\sigma_1$ and $\sigma_2$ are plotted on $X$ and $Y$ axes respectively. Tensile stresses are considered as positive,

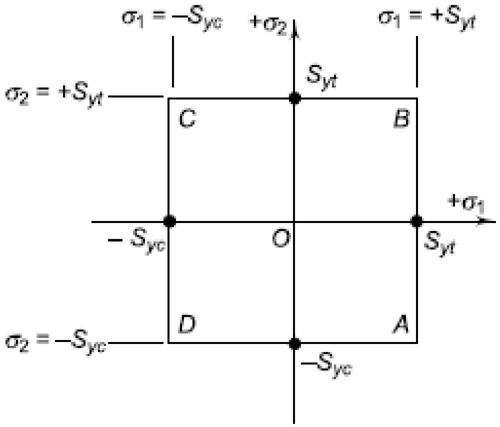


**Fig. 4.31** *Boundary for Maximum Principal Stress Theory under Bi-axial Stresses*

while compressive stresses as negative. It is further assumed that *Syc* = *Syt,* It should be noted that,

(i) The equation of vertical line to the positive side of *X*-axis is ( $x = + a$)

(ii) The equation of vertical line to the negative side of *X*-axis is ( $x = - a$)

(iii) The equation of horizontal line to the positive side of *Y*-axis is ( $y = + b$)

(iv) The equation of horizontal line to the negative side of *Y*-axis is ( $y = - b$)

The borderline for the region of safety for this theory can be constructed in the following way:

***Step 1: Suppose*** $\sigma_1 > \sigma_2$ As per this theory, we will consider only the maximum of principal stresses ($\sigma_1$) and disregard the other principal stress ($\sigma_2$).

Suppose ($\sigma_1$) is the tensile stress. The limiting value of ($\sigma_1$) is yield stress ($S_{yt}$). Therefore, the boundary line will be,

$$\sigma_1 = + S_{yt}$$

A vertical line $\overline{AB}$ is constructed such that $\sigma_1 = + S_{yt}$.

*Step 2:* Suppose $\sigma_1 > \sigma_2$ and ($\sigma_1$) is compressive stress. The limiting value of ($\sigma_1$) is compressive yield stress ($- S_{yc}$). Therefore, the boundary line will be,

$$\sigma_1 = - S_{yc}$$

A vertical line $\overline{DC}$ is constructed such that $\sigma_1 = - S_{yc}$.

***Step 3: Suppose*** $\sigma_2 > \sigma_1$ As per this theory, we will consider only the maximum of principal stresses ($\sigma_2$) and disregard the other principal stress ($\sigma_1$).

Suppose ($\sigma_2$) is the tensile stress. The limiting value of ($\sigma_2$) is the yield stress ($S_{yt}$). Therefore, the boundary line will be,

$$\sigma_2 = + S_{yt}$$

A horizontal line $\overline{CB}$ is constructed such that $\sigma_2 = + S_{yt}$

*Step 4:* Suppose $\sigma_2 > \sigma_1$ and $(\sigma_2)$ is the compressive stress. The limiting value of $(\sigma_2)$ is compressive yield stress $(-S_{yc})$. Therefore, the boundary line will be,

$$\sigma_2 = -S_{yc}$$

A horizontal line $\overline{DA}$ is constructed such that $\sigma_2 = -S_{yc}$.

The complete region of safety is the area *ABCD*. Since we have assumed $(S_{yc} = S_{yt})$, *ABCD* is a square.

MAXIMUM SHEAR STRESS THEORY

This criterion of failure is accredited to CA Coulomb, H Tresca and JJ Guest. The theory states that the failure of a mechanical component subjected to bi-axial or tri-axial stresses occurs when the maximum shear stress at any point in the component becomes equal to the maximum shear stress in the standard specimen of the tension test, when yielding starts. In the tension test, the specimen is subjected to uni-axial stress (σ1) and (σ2 = 0). The stress in the specimen of tension test and the corresponding Mohr's circle diagram are shown in Fig. 4.32. From the fi gure,

$$\tau_{max} = \frac{\sigma_1}{2}$$

When the specimen starts yielding $(\sigma_1 = S_{yt})$, the above equation is written as

$$\tau_{max} = \frac{S_{yt}}{2}$$

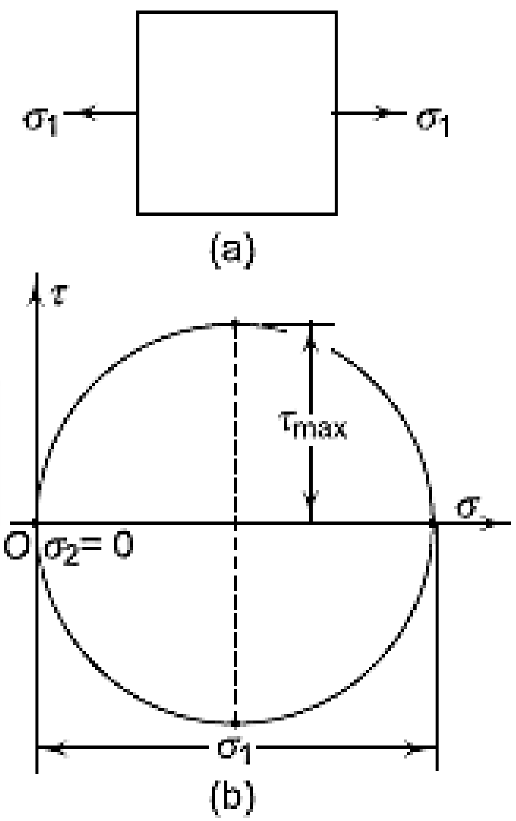


**Fig. 4.32** *(a) Stresses in Simple Tension Test*
*(b) Mohr's Circle for Stresses*