Therefore, the maximum shear stress theory predicts that the yield strength in shear is half of the yield strength in tension, i.e.,

$$S_{sy} = 0.5\, S_{yt} \qquad (4.38)$$

Suppose σ_1, σ_2 and σ_3 are the three principal stresses at a point on the component, the shear stresses on three different planes are given by,

$$\tau_{12} = \left(\frac{\sigma_1 - \sigma_2}{2}\right) \quad \tau_{23} = \left(\frac{\sigma_2 - \sigma_3}{2}\right)$$

$$\tau_{31} = \left(\frac{\sigma_3 - \sigma_1}{2}\right) \qquad (a)$$

The largest of these stresses is equated to (τ_{max}) or $(S_{yt}/2)$.

Considering factor of safety,

$$\left(\frac{\sigma_1 - \sigma_2}{2}\right) = \frac{S_{yt}}{2\,(fs)}$$

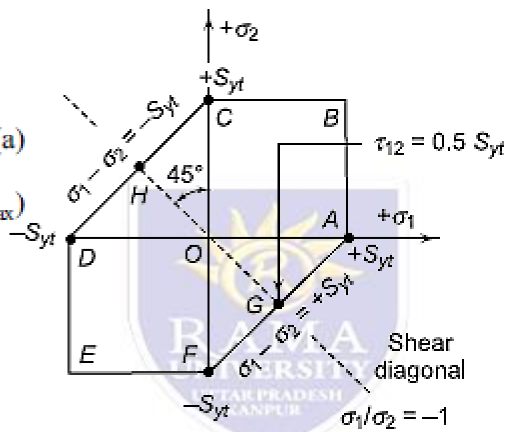
or $$(\sigma_1 - \sigma_2) = \frac{S_{yt}}{(fs)}$$

$$(\sigma_2 - \sigma_3) = \frac{S_{yt}}{(fs)}$$

$$(\sigma_3 - \sigma_1) = \frac{S_{yt}}{(fs)} \qquad (4.39)$$

The above relationships are used to determine the dimensions of the component. Refer to expression (a) again and equating the largest shear stress (τ_{max}) to $(S_{yt}/2)$,

$$\left(\frac{\sigma_1 - \sigma_2}{2}\right) = \frac{S_{yt}}{2}$$



.33 *Boundary for Maximum Shear Stress Theory under Bi-axial Stresses*

or $$\sigma_1 - \sigma_2 = S_{yt} \qquad (b)$$

Similarly,

$$\sigma_2 - \sigma_3 = S_{yt} \qquad (c)$$

$$\sigma_3 - \sigma_1 = S_{yt} \qquad (d)$$

For compressive stresses,

$$\sigma_1 - \sigma_2 = -S_{yc} \qquad (e)$$

$$\sigma_2 - \sigma_3 = -S_{yc} \qquad (f)$$

$$\sigma_3 - \sigma_1 = -S_{yc} \qquad (g)$$

The above equations can be written as,

$$\sigma_1 - \sigma_2 = \pm S_{yt} \qquad [\text{Assuming } S_{yc} = S_{yt}]$$

$$\sigma_2 - \sigma_3 = \pm S_{yc}$$

$$\sigma_3 - \sigma_1 = \pm S_{yt}$$

Region of Safety For bi-axial stresses,

$$\sigma_3 = 0$$

The above equations can be written as,

$$\sigma_1 - \sigma_2 = \pm S_{yt} \qquad (h)$$

$$\sigma_2 = \pm S_{yt} \qquad (i)$$

$$\sigma_1 = \pm S_{yt} \qquad (j)$$

It will be observed at a later stage that Eq. (h) are applicable in second and fourth quadrants, while Eqs (i) and (j) are applicable in the first and third quadrants of the diagram.

The construction of region of safety is illustrated in Fig. 4.33. The two principal stresses σ_1 and σ_2 are plotted on the X and Y axes respectively. Tensile stresses are considered as positive, while compressive stresses as negative.

- It should be noted that,
- (i) The equation (x – y = – a) indicates a
- straight line in the second quadrant with (–a)
- and (+ a) as intercepts on the X and Y axes
- respectively.
- (ii) The equation (x – y = + a) indicates a
- straight line in the fourth quadrant with (+ a)
- and (– a) as intercepts on the X and Y axes
- respectively.
- The borderline for the region of safety for this
- theory can be constructed in the following way:

Step 1: In the first quadrant, both (σ_1) and (σ_2) are positive or tensile stresses. The yielding will depend upon where (σ_1) or (σ_2) is greater in magnitude.

Suppose $\sigma_1 > \sigma_2$

The boundary line will be,

$$\sigma_1 = + S_{yt}$$

A vertical line $\overline{AB}$ is constructed such that

$$\sigma_1 = + S_{yt}.$$

Suppose $\sigma_2 > \sigma_1$

The boundary line will be,

$$\sigma_2 = + S_{yt}$$

A horizontal line $\overline{CB}$ is constructed such that $\sigma_2 = + S_{yt}$

Step 2: In the third quadrant, both (σ_1) and (σ_2) are negative or compressive stresses. The yielding will depend upon whether (σ_1) or (σ_2) is greater in magnitude.

Suppose $\sigma_1 > \sigma_2$

The boundary line will be,

$$\sigma_1 = - S_{yt}$$

A vertical line $\overline{DE}$ is constructed such that

$$\sigma_1 = - S_{yt}$$

Suppose $\sigma_2 > \sigma_1$

The boundary line will be,

$$\sigma_2 = - S_{yt}$$

A horizontal line $\overline{EF}$ is constructed such that

$$\sigma_2 = - S_{yt}$$

Step 3: In the second and fourth quadrants, (σ_1) and (σ_2) are of opposite sign. One stress is tensile while the other is compressive. The yielding will occur when,

$$\sigma_1 - \sigma_2 = \pm S_{yt}$$

In the second quadrant, line $\overline{DC}$ is constructed such that,

$$\sigma_1 - \sigma_2 = - S_{yt}$$

It is observed that the intercept of the above line on the *X*-axis ($\sigma_2 = 0$) is ($- S_{yt}$) and intercept on the *Y*-axis ($\sigma_1 = 0$) is ($+ S_{yt}$).

Step 4: In the fourth quadrant, line $\overline{FA}$ is constructed such that,

$$\sigma_1 - \sigma_2 = +S_{yt}$$

It is observed that the intercept of the above line on the X-axis ($\sigma_2 = 0$) is ($+S_{yt}$) and intercept on the Y-axis ($\sigma_1 = 0$) is ($-S_{yt}$).

The complete region of safety is the hexagon $ABCDEFA$.

In case of bi-axial stress, if a point with coordinates (σ_1, σ_2) falls outside this hexagon region, then it indicates the failure condition. On the other hand, if the point falls inside the hexagon, the design is safe and the failure may not occur.

Shear Diagonal *Shear diagonal or line of pure shear is the locus of all points, corresponding to pure shear stress.* It will be proved at a later stage (Fig. 4.35) that for pure shear stress,

$$\sigma_1 = -\sigma_2 = \tau_{12}$$

The above equation can be written as,

$$\frac{\sigma_1}{\sigma_2} = -1 = -\tan(45°)$$

A line $\overline{GH}$ is constructed in such a way that it passes through the origin O and makes an angle of $-45°$ with the Y-axis. This line is called shear diagonal or line of pure shear. This line intersects the hexagon at two points G and H. The point of intersection of lines $\overline{FA}$ ($\sigma_1 - \sigma_2 = +S_{yt}$) and $\overline{GH}$ $\left[\frac{\sigma_1}{\sigma_2} = -1\right]$ is G.

Solving two equations simultaneously,

$$\sigma_1 = -\sigma_2 = +S_{yt}/2$$

Since $\quad \sigma_1 = -\sigma_2 = \tau_{12}$

$$\tau_{12} = \frac{1}{2} S_{yt}$$

Since the point G is on the borderline, this is the limiting value for shear stress.

or $$S_{sy} = \frac{1}{2} S_{yt}$$

The maximum shear stress theory of failure is widely used by designers for predicting the failure of components, which are made of ductile materials, like transmission shaft.

- DISTORTION-ENERGY THEORY
- This theory was advanced by MT Huber in Poland(1904) and independently by R von Mises in Germany (1913) and H Hencky (1925). It is known as the Huber von Mises and Hencky's theory. The theory states that the failure of the mechanical component subjected to bi-axial or tri-axial stresses occurs when the strain energy of distortion per unit volume at any point in the component, becomes equal to the strain energy of distortion per unit volume in the standard specimen of tension-test, when yielding starts.

A unit cube subjected to the three principal stresses σ_1, σ_2 and σ_3 is shown in Fig. 4.34(a). The total strain energy U of the cube is given by,

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3 \qquad \text{(a)}$$

where ε_1, ε_2 and ε_3 are strains in respective directions.

Also, $$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \mu(\sigma_1 + \sigma_3)]$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)] \qquad \text{(b)}$$

Substituting the above expressions in Eq. (a),

$$U = \frac{1}{2E}[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$
$$-2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \qquad \text{(c)}$$

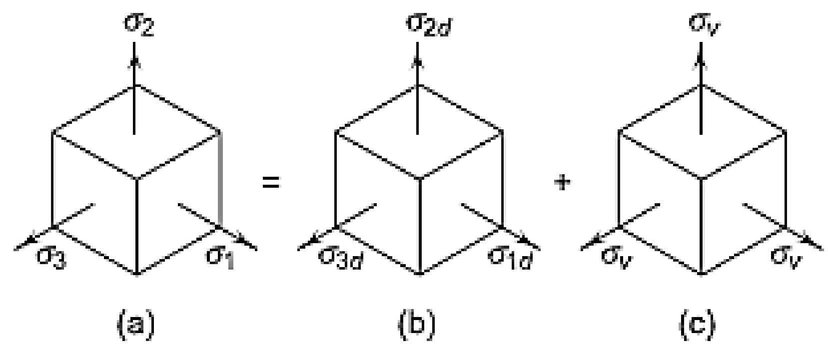


Fig. 4.34 *(a) Element with Tri-axial Stresses (b) Stress Components due to Distortion of Element (c) Stress Components due to Change of Volume*

The total strain energy U is resolved into two components—fi rst Uv corresponding to the change of volume with no distortion of the element and the second Ud corresponding to the distortion of the element with no change of volume. Therefore,

U = Uv + Ud (d)

The corresponding stresses are also resolved into two components as shown in Fig. 4.34 (b) and (c). From the fi gure

$$\sigma_1 = \sigma_{1d} + \sigma_v$$
$$\sigma_2 = \sigma_{2d} = \sigma_v$$
$$\sigma_3 = \sigma_{3d} = \sigma_v$$

The components σ_{1d}, σ_{2d} and σ_{3d} cause distortion of the cube, while the component σ_v results in volumetric change. Since the components σ_{1d}, σ_{2d} and σ_{3d} do not change the volume of the cube,

$$\varepsilon_{1d} + \varepsilon_{2d} + \varepsilon_{3d} = 0 \qquad \text{(f)}$$

Also,

$$\varepsilon_{1d} = \frac{1}{E}[\sigma_{1d} - \mu(\sigma_{2d} + \sigma_{3d})]$$

$$\varepsilon_{2d} = \frac{1}{E}[\sigma_{2d} - \mu(\sigma_{1d} + \sigma_{3d})]$$

$$\varepsilon_{3d} = \frac{1}{E}[\sigma_{3d} - \mu(\sigma_{1d} + \sigma_{2d})] \qquad \text{(g)}$$

Substituting Eq. (g) in Eq. (f),

$$(1 - 2\mu)\,(\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) = 0$$

Since $\quad (1 - 2\mu) \neq 0$

$$\therefore \qquad (\sigma_{1d} + \sigma_{2d} + \sigma_{3d}) = 0 \qquad \text{(h)}$$

From Eq. (h) in Eq. (e),

$$\sigma_v = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad \text{(j)}$$

The strain energy U_v corresponding to the change of volume for the cube is given by,

$$U_v = 3\left[\frac{\sigma_v \varepsilon_v}{2}\right] \qquad \text{(k)}$$

Also $\qquad \varepsilon_v = \frac{1}{E}[\sigma_v - \mu(\sigma_v + \sigma_v)]$

or $\qquad \varepsilon_v = \frac{(1 - 2\mu)\sigma_v}{E} \qquad$ (l)

From expressions (k) and (l),

$$U_v = \frac{3(1 - 2\mu)\sigma_v^2}{2E} \qquad \text{(m)}$$

Substituting expression (j) in the Eq. (m),

$$U_v = \frac{(1 - 2\mu)\,(\sigma_1 + \sigma_2 + \sigma_3)^2}{6E} \qquad \text{(n)}$$

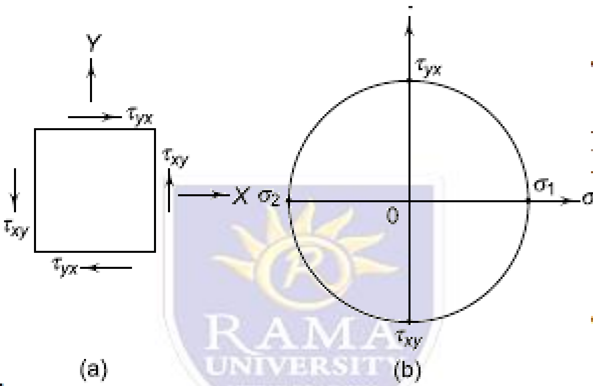


Fig. 4.35 *(a) Element subjected to Pure Shear Stresses (b) Mohr's Circle for Shear Stresses*

From expressions (c) and (n),

$$U_d = U - U_v$$

or $\qquad U_d = \left(\frac{1 + \mu}{6E}\right)[(\sigma_1 - \sigma_2)^2$

$$+ (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \qquad (4.40)$$

In simple tension test, when the specimen starts yielding,

$$\sigma_1 = S_{yt} \qquad \text{and} \qquad \sigma_2 = \sigma_3 = 0$$

Therefore, $\qquad U_d = \left(\frac{1 + \mu}{3E}\right) S_{yt}^2 \qquad$ (4.41)

From Eqs (4.40) and (4.41), the criterion of failure for the distortion energy theory is expressed as

$$2S_{yt}^2 = \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

or $\qquad S_{yt} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \qquad$ (4.42)

Considering the factor of safety,

$$\frac{S_{yt}}{(fs)} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \qquad (4.43)$$

For bi-axial stresses ($\sigma_3 = 0$),

$$\frac{S_{yt}}{(fs)} = \sqrt{(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)} \qquad (4.44)$$

A component subjected to pure shear stresses and the corresponding Mohr's circle diagram is shown in Fig. 4.35.